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Discussion on: '(A, B)-Invariance Conditions of Polyhedral Domains for Continuous-Time Systems' by C.E.T. Dorea and J.-C. Hennet

Denoting by I the vector I = [1 1... 1] then

5 = {x =Xa a 2: 0, fa = I}

In the discrete-time case

x(k + 1) =Ax(k) + Bu(k)

5 is (A, B)-invariant if and only if there exists a matrix U and a non-negative matrix P of appropriate dimensions such that

This condition can be derived by the vertices con-ditions of Gutman and Cwikel [2], which say that 5 is (A, B)-invariant iff for every vertex Xj there exists a control Uj such that AXj + BUj E S. These conditions can be written as

1. Discussion by F. BlanchinF and S. MianF

This part of the discussion will consider the follow-ing issues:

1. dual conditions for (A, B)-invariance;
2. infinite directions: stability and minimum-phase conditions;
3. bi-linearity of the (A, B)-invariance conditions when the region is not fixed;
4. robustness issues;
5. linear vs. non-linear control laws;
6. summary.

1.1. The Dual Conditions

AX + BU =XP, fp:5 f (1)

Axj + BUj =2: xiPij
i=I

With 'ii=lPij:5 1, and Pij 2: 0 and are equivalent to the matrix conditions above. A similar condition holds in the continuous-time case. A matrix P is

said M matrix (see [3]) iff all its non-diagonal entries are non-negative. Then 5 is (A, B)-invariant iff there exists U and an J\1-matrix M such that

One simple way to prove it is to use Proposition 4.1 of the paper noting that the existence of 'T > o such that 5 is (A, B)-invariant for the Euler Approximating System is equivalent to the invari-ance of the continuous-time system [4].
The conditions (1) and (2) apply under control constraints, because U is the control-at-vertices

AX + BU = XM, fM:5 0 (2)

The proposed conditions (7)-(9) of the paper con-sider the 'plane' representation of a polyhedral set 5 = R[G, p] and are both necessary and sufficient for its (A, B)-invariance.
There exist dual conditions which involve the

vertex representation of 5 as long as such a set is compact and includes the origin [1]. Assume that X is the matrix whose columns Xi are the vertices of 5:

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Discussion on '(A, B)-Invariance Conditions of Polyhedral Domains for Continuous-Time Systems' 83

matrix and thus it is sufficient to add Uk E V if V
is the control constraint set. Note also that as long
as S includes the.origin in its interior contraction
parameters 0 < A < 1 and E > 0 can be included
in the invariance condition, to assure convergence
to the origin, by replacing the inequalities in (1) and (2) by 1P:::; Al and 1M:::; -EI, for the discrete and continuous-time respectively.
These conditions require the vertex representation of S, which is less natural than the plane represen-tation. Note that we can always pass from a plane representation to a vertex but this transformation usually leads to a great increase in complexity. On the other hand, the plane conditions proposed in the paper require evaluation of the generator of the polyhedral cone r in Eq. (6) of the paper, whose computation complexity is comparable to that of the computation of the vertices.
We argue that the aforementioned dual conditions can be extended to unbounded sets by using the idea of 'infinite directions' introduced in the paper.

1.2. Infinite Directions: Stability and Minimum-Phase Conditions

ping criteria and provide invariant sets which are maximal inside given constraint regions. However, no one knows a priori how many planes/vertices are going to produce when they stop.
For ellipsoidal invariant sets there exist methods based on convex optimisation [9] which allow for the determination of invariant sets satisfying some specifications such as the largest (A, B)-invariant ellipsoid inside a polytope or the smallest (A, B)- invariant ellipsoid including a polytope. However, these regions are conservative in general. For instance, the largest invariant ellipsoid included in a polytope can be considerably smaller than the actual largest invariant domain, whereas the latter can always be approximated by an invariant polyhedron.
It would be very important to find procedures based on conditions such as (7)-(9), or those mentioned above, for the efficient computation of polyhedral invariant sets (given appropriate specifications) with G, p or XV of assigned dimen-sions.

1.4. Robustness Issues

x(t) = A(t)x(t) + B(t)u(x(t))

A(t) =2: A;w;(t), B(t) =2: B;w;(t)

The conditions given in the paper, as well as the dual conditions we mentioned in Section 1.1 [1], can be immediately extended to the case with parametric uncertainties:

A question concerning the infinite directions arises from this paper and a previous paper by one of the authors [5]. For the convenience of the exposition let us consider the symmetric case

S = {x: - p:::; Gx:::; p}

As long as the set S is not compact, the infinite directions are associated to some (A, B)-invariant sub-spaces. The non-compactness of S leads to some questions concerning the closed-loop system stability. Have some minimum-phase conditions to be imposed of the triple (A, B, G) (possibly with rank conditions on the matrix GB) as was the case in Castelan and Hennet [5]?

;=1

where

2: w;:::; I and w;:5 O.
;=1

i=1

1.3. Bi-linearity of the (A, B)-Invariance Conditions when the Region S is not Fixed

The main problem of these approaches when the (A, B)-invariant set has to be constructed is that the basic conditions are non-linear as long as the region S is not fixed, but G and p have to be determined. This fact is true for both the mentioned 'vertex' or 'plane' conditions. The only effective construction techniques, currently known, involve iterative methods for the construction of invariant sets.

Examples of such procedures can be found else-where [2,4,6-8]. These procedures have finite stop-

Indeed, in this case, the (A, B)-invariance of S (i.e., the existence of a continuous control u(x) such that S is robustly positively invariant for the closed-loop system) is equivalent to the existence of a single continuous control u(x) such that the set is positively invariant for all the subsystems:

x(t) =A;X(t) + B;u(t)

The condition provided in the paper can then be easily restated by involving a set of matrices Yi) and T;) i = 1, "', s for which (7)-(9) hold for each pair (A;, B;).

84 Discussion on '(A, B)-Invariance Conditions of Polyhedral Domains for Continuous-Time Systems'

1.5. Linear vs. Non-linear Control Laws

The fact that the region S has a very complex
representation is not in principle a problem for its
determination. Indeed, its computation is performed
off-line where there are no hard constraints on the
computation time.
The actual problem is due to the fact that feed-back control is based on such a region. Theorem 5.1 reports sufficient conditions for the existence of a linear feedback gain (plus an eventual constant term uc

) guaranteeing (A, B)-invariance of an assigned region. Unfortunately, these conditions are quite strong and moreover they are not very effective for synthesis purposes, so for example it is not possible to determine an (A, B)-invariant region (subject to some specification) admitting a stabilising linear feedback. Recall that (A, B)-invariant ellip-soids can be always associated to a linear controller. Since the polyhedral sets have several well-known advantages over ellipsoids, we wonder if the pro-posed conditions could be eventually used in a constructive way for the construction of (A, B)- invariant polyhedra associated to a linear compen-
sator.

Thus, up to now the only applicable control strategy is the non-linear Gutman and Cwikel con-trol, eventually extended to unbounded sets to the light of the proposed results, which requires vertex/infinite-directions computation and partition of the polytope into simplicial sectors whose number can be very high. This is one of the main drawbacks of the approach based on non-linear controllers. An alleviation of the computational burden can be obtained by smoothing the polyhedral Lyapunov function associated to the (A, B)-invariant domain (see BIanchini and Miani [10] for details).

1.6. Summary

The new (A, B)-invariance conditions based on the hyperplane description of the polyhedral region com-plete in a dual sense the existing ones. However, the main problems arising when dealing with these regions, for example the controller/region com-plexity, remain open and further work in this area is needed.

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2. Final Comments by the Authors C.E.T. Dorea3 and J.-C. Hennet4

In the field of linear systems, research on invariance conditions for polyhedral domains has been very active during the last decade. However, the dis-cussion paper by F. BIanchini and S. Miani shows that the subject is still far from having been fully explored.
The dual conditions (1) and (2) of the discussion paper provide an alternative characterisation of (A, B)-invariance of a polyhedral domain. Condition (1), which applies to the discrete-time case, is a nice analytical formulation of the results of Gutman and Cwikel [1]. And condition (2) can be derived from (1), using the result by BIanchini [2] which estab-lishes the equivalence of invariance of polyhedral domains with respect to a continuous-time linear system and with respect to some of its possible Euler approximating systems (EAS).
It can be noted that the statement of this invari-

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Discussion on '(A, B)-Invariance Conditions of Polyhedral Domains for Continuous-Time Systems' 85

where v is a row vector and w a non-negative scalar.
The pre-conditions on (v, w) can be rewritten

1. all its off-diagonal entries are non-positive;
2. - AfT is diagonal dominant, since - AfT 1 2: 0
and - AfT 1 O.

tGAx::; tp Vt 2: 0 such that tGB =QT,

Vx:Gx 2: p (3)

Yp::; Tp (5)

which can be rewritten as

(l + TA)X + TBU = XP, 1Tp::; 1T, with P 2: 0

By application of Farkas' lemma as in the proof of the discrete-time case, one can conclude that (A, B)- invariance of R[G, p] is equivalent to the existence of a non-negative matrix L such that

1 1
LG =- TG(l + TA), Lp::; - Tp, with L 2: 0
T T

where the row vectors of matrix T form a set of

generators of the polyhedral cone r, which is the non-negative left kernel of GB.
The same proof applies to the continuous-time case. One has just to notice that there always exists a non-negative matrix P and a positive scalar T such that an essentially non-negative matrix M can be written in the form: M = 1h(P - l). Therefore, condition (2) of the discussion paper can be equival-ently written in the following form:

which corresponds to the conditions of Theorem 3.1, with Y = (L - 1hY).
The preceding proof is also useful to clarify the relation between the original continuous-time system and discrete-time EAS.
Concerning the complexity associated with the computation of the generators of the polyhedral cone r, it depends mostly on the number of control entries. In particular, for single-input systems, the computation of matrix T via the Fourier-Motzkin elimination technique practically reduces to the classification of the signs of the elements of the column matrix GB, regardless of the complexity of the polyhedron.
In the light of the comments by F. BIanchini and S. Miani on stability, minimum-phase conditions and robustness issues, it is worth stressing the fact that our paper has been essentially focused on (A, B)- invariance conditions. In practice, invariance of a well-chosen polyhedral domain is often set as a constraint or an objective to be achieved by a controller, in order to guarantee stability, perform-
ance and robustness. In particular, such a use of polyhedral invariance conditions has been shown to
be appropriate in the II control problem for the discrete-time case [6,7] and in the L' control prob-
lem for the continous-time case [8].

(1)

(2)

(4)

vAX::; wI T V(v, w) such that vX::; wI T and

YG= TGA

w 2: 0, vB =OT with OT =[0... 0]

vB =OT, vx::; w Vx: Gx 2: p

Again by Farkas' lemma, existence of (v, w) satisfying (2) is equivalent to the existence of a non-negative vector t such that v = tG and tp::; w. Replacing v by tG in condition (2) and noting that tp is the minimal possible value of w, condition (2) can be reformulated in the form

ance result for the continuous-time case could be modified using a more classical definition of M-matrices, as can be found in Gantmacher [3] or in Berman and Plemmons [4]. In their terminology, it is the opposite of matrix M of condition (2), - M, which is an M-matrix because

The following (A, B)-invariance conditions for the discrete-time case can then be derived from (3) as in Dorea and Hennet [5]. (A, B)-invariance of 5 is equivalent to the existence of a non-negative matrix Y such that

In agreement with this terminology, a matrix having all its non-diagonal entries non-negative is called an essentially non-negative matrix [4].
Some results of our paper can be obtained from the (A, B)-invariance conditions proposed by F. BIanchini and S. Miani. Further results might also be obtained using the complementarity of the two dual sets of conditions.
As can be expected from duality, the (A, B)- invariance conditions of our paper can be derived from the dual conditions (1), (2) in F. BIanchini and S. Miani. Such a derivation provides an alterna-tive proof to our Theorem 3.1.
Consider first the discrete-time case.
By application of Farkas' lemma to condition (1) in F. BIanchini and S. Miani, for all the vertices, (A, B)-invariance of the polyhedral domain 5 is equivalent to the following condition:

86 Discussion on '(A, B)-lnvariance Conditions of Polyhedral Domains for Continuous-Time Systems'

Concerning stability, it has been shown [9] that a
necessary condition for a symmetrical polhyedron S =

{x: IGxl:::: p} to be (A, B)-invariant is that:Ker(G)
be an (A, B)-invariant subspace. Therefore, positive
invariance with closed-loop stability of an (A, B)-
invariant unbounded symmetrical polyhedron S is achi-evable only if:Ker(G) is an internally stabilisable (A, B)-invariant subspace. This clearly requires, if the triplet (A, B, G) had invariant zeros, that they belong to the stability region of the complex plane.
It is true that the effectiveness of the sufficient conditions for existence of a linear control law, established in Theorem 5.1 of the paper, is limited by the fact that they can only be checked a pos-teriori. However, we have observed through a large number of experiments that they often happen to be satisfied when the polyhedron is (A, B)-invariant. Moreover, they can be checked very simply. We therefore argue that one should use Theorem 5.1, to check for existence of a linear control law, before undertaking the computation of the simplicial regions associated with a piecewise-linear control law. We agree, however, on the fact that computation of an effective control law in the gen-eral case remains an open problem, further work being needed in this direction.

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